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On Multiple-Shaker Resonance Testing

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A simulation study is carried out to explore the use of vector response data from single-shaker modal survey tests in tuning the shaker inputs for multiple-shaker tests. A modal purity criterion based on phase coherence, i.e., displacements should lag excitation by 90°, is defined. A "test natural frequency" is defined as any excitation frequency which produces, at all excitation stations, response in quadrature with the excitation forces. A "test natural mode" is defined as the quadrature response at all measurement stations when the system is excited at a test natural frequency. Calculations based on single-shaker response data determine "test natural frequencies," "test natural modes," and the required shaker force ratios for multiple-shaker tests. These frequencies and modes are compared with exact ones for two nine-degree-of-freedom damped systems, one having two frequencies very closely spaced. It is shown that both acceptable and spurious test natural frequencies are obtained, but that acceptable ones persist (i.e., occur for most possible shaker combinations). "Test natural modes" corresponding to spurious frequencies have poor phase coherence at nonexcited stations and can thus be readily identified. Acceptable mode shape information can be obtained even when frequencies are closely spaced.

Nomenclature

$[\bar{B}]$ = admittance matrix for all n coordinates
$[\bar{B}_*]$ = admittance matrix for p coordinates
$[C]$ = damping coefficient matrix
$\{F\}$ = input force vector at n stations
$\{F_*\}$ = input force vector at p stations
$[K]$ = stiffness matrix
$[M]$ = mass matrix
n = number of degrees of freedom, total number of displacement measurement stations
$\{p\}$ = principal coordinates
p = number of shakers used
$\{\bar{u}\}$ = harmonic response = $\{\bar{U}\} e^{i\omega t}$
$\{v\}$ = original displacement coordinates
$\{V\}$ = response amplitude vector
$[V]$ = modal matrix
$[\gamma]$ = damping matrix in principal coordinates
ε = allowable phase error
θ = phase lag angle
$[\kappa]$ = stiffness matrix in principal coordinates
$[\mu]$ = mass matrix in principal coordinates
$\{\Phi\}$ = force vector in principal coordinates
ω, ω_r = excitation frequency (rad/sec), rth natural frequency

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Introduction

METHODS for experimentally determining the dynamical characteristics of structures using resonance, or modal survey, testing may be classified as single-shaker or multiple-shaker methods. The latter category includes the possibility of using information from one or more single-shaker tests in setting up multiple-shaker tests. The purpose of resonance testing is to determine information on one or more of the following dynamical characteristics of the system: natural frequencies, mode shapes, damping factors, and generalized masses.

Kennedy and Pancu[1] suggested that certain characteristics of vector response plots obtained with single-shaker excitation could be used in determining the natural frequencies and damping factors of systems. Bishop and Gladwell[2] and Pendered[3] assessed the accuracy of several single-shaker techniques, including the method of Kennedy and Pancu. It may be said that, if properly interpreted, natural frequencies obtained by the method of Kennedy and Pancu are generally acceptable, but mode-shapes results are frequently unacceptable.

North and Stephenson[4] reported on the ambitious multiple-shaker tests of the XB-70 airplane, and multiple-shaker testing has also been discussed by other authors.[5-7] Smith and Woods[8] conducted multiple-shaker resonance tests on a small model to demonstrate an energy technique. Most recently, modal survey testing employing up to twelve shakers simultaneously was conducted at NASA Manned Spacecraft Center on the manned orbital lab.[9] Multiple-shaker testing is complex and costly, and the results obtained depend upon how successfully the "tuning" of modes has been accomplished. This success, as well as the success of single-shaker testing, is strongly influenced by the closeness of the frequency spacing. The effect of closely-spaced frequencies on resonance test results has been considered by Pendered,[3] Traill-Nash et al.[10] and Craig.[11]

The purpose of the present paper is to explore the following questions: 1) What are appropriate criteria for judging the accuracy of natural frequency and mode-shape information obtained from multiple-shaker tests? 2) How many shakers are required to excite a mode which satisfies the specified criteria? 3) What guidelines should be used in selecting locations for additional shakers? 4) What effect does close frequency spacing have on the testing procedures? The simulation study reported on in the present paper was conducted on the digital computer. However, it is an extension of work done with a physical model and with analog computer models and reported on in Ref. 11.

The theory of resonance testing is briefly reviewed in order that modal purity criteria can be defined precisely. The number of shakers required to excite a mode which satisfies the specified criteria is then discussed. A procedure suggested by Asher[12] for determining the test natural frequencies and corresponding shaker force ratios is explored. Through the use of a simulation study many significant features of Asher's method are clarified and new features are brought to attention. Based on the results of the simulation study and the present level of sophistication of modal survey test equipment, a procedure for employing Asher's method for tuning of modes is suggested.

Resonance Testing Theory

The mathematical model generally used to describe harmonic motion of an aerospace structure is that of an n-degree-of-freedom, linear, damped system described by the equation of motion

$$[M]\{\ddot{v}\}+(1/\omega)[C]\{\dot{v}\}+[K]\{v\} = \{F\}\sin\omega t \quad (1)$$

In this paper it is assumed that the excitation forces are all in phase, i.e., have either 0° or 180° phase relative to some reference force. It is convenient to introduce the complex form of Eq. (1), namely

$$[M]\{\ddot{\bar{u}}\}+(1/\omega)[C]\{\dot{\bar{u}}\}+[K]\{\bar{u}\} = \{F\}e^{i\omega t} \quad (2)$$

where

$$\{v\} = Im\{\bar{u}\} \quad (3)$$

Complex quantities are denoted by a bar. If the amplitude of $\{\bar{u}\}$ is $\{\bar{U}\}$, then Eq. (2) gives

$$([K]-\omega^2[M]+i[C])\{\bar{U}\} = \{F\} \quad (4)$$

which may be written

$$\{\bar{U}\} = [\bar{B}]\{F\} \quad (5)$$

The matrix $[\bar{B}]$ is called the complex admittance matrix. Its real and imaginary parts will be defined by

$$[\bar{B}] = [B']+i[B''] \quad (6)$$

The calculation of $[B']$ and $[B'']$ for the general case is somewhat involved, but for the special case when damping does not couple the modes the calculation of $[B']$ and $[B'']$ is straightforward. Since this case is almost universally assumed in theoretical studies as well as in models used for interpreting experimental results, it will be employed in the simulation study reported here.

Through the transformation

$$\{\bar{u}\} = [V]\{\bar{p}\} \quad (7)$$

where $[V]$ contains the free-vibration modes (eigenvectors) as columns, Eq. (2) may be written in principal coordinates as

$$[\mu]\{\ddot{\bar{p}}\}+(1/\omega)[\gamma]\{\dot{\bar{p}}\}+[\kappa]\{\bar{p}\} = \{\Phi\}e^{i\omega t} \quad (8)$$

where

$$\begin{aligned}[\mu] &= [V]^T[M][V] = \mathrm{diag}(\mu_1,\mu_2,\ldots,\mu_n)\\ [\gamma] &= [V]^T[C][V]\\ [\kappa] &= [V]^T[K][V] = \mathrm{diag}(\kappa_1,\kappa_2,\ldots,\kappa_n)\end{aligned} \quad (9)$$

and

$$\{\Phi\} = [V]^T\{F\}$$

In cases where the damping $[C]$ does not couple the equations in principal coordinates, $[\gamma]$ is diagonal and will be denoted by

$$[\gamma] = \mathrm{diag}(\gamma_1,\gamma_2,\ldots,\gamma_n) \quad (10)$$

Then the solution of Eq. (8) may be written as

$$\bar{p}_r = \frac{\Phi_r e^{i(\omega t-\eta_r)}}{[\gamma_r^2+(\kappa_r-\omega^2\mu_r)^2]^{1/2}} \quad (11)$$

where

$$\tan\eta_r = \gamma_r/(\kappa_r-\omega^2\mu_r) \quad (12)$$

Combining Eqs. (11, 9d, and 7) gives

$$\bar{u}_i = \sum_{r=1}^{n}\sum_{j=1}^{n}\frac{V_{ir}V_{jr}F_j e^{i(\omega t-\eta_r)}}{[\gamma_r^2+(\kappa_r-\omega^2\mu_r)^2]^{1/2}} \quad (13)$$

Thus, the element $\bar{b}_{ij}$ of the complex admittance matrix is given by

$$\bar{b}_{ij} = \sum_{r=1}^{n}\frac{V_{ir}V_{jr}e^{-i\eta_r}}{[\gamma_r^2+(\kappa_r-\omega^2\mu_r)^2]^{1/2}} \quad (14)$$

Although the complex admittance matrix is not so simply obtained for a system with damping coupling, an important special result concerning the forced response of a coupled system will now be obtained. Consider again Eq. (1). The solution vector $\{v\}$ has elements which may be written

$$v_i = V_i\sin(\omega t-\theta_i) \quad (15)$$

where V_i is the (unknown) amplitude and θ_i is the phase lag relative to the reference force. Bishop and Gladwell[2] have shown that there exist certain forcing vectors $\{F\}$ such that all v_i are in phase with each other at a phase angle θ relative to the excitation. Then the vector $\{v\}$ will have the form

$$\{v\} = \{V\}\sin(\omega t-\theta) \quad (16)$$

Eqs. (1) and (16) may be combined to give

$$\begin{aligned}-\sin\theta([K]-\omega^2[M])\{V\}+\cos\theta[C]\{V\} &= \{0\}\\ \cos\theta([K]-\omega^2[M])\{V\}+\sin\theta[C]\{V\} &= \{F\}\end{aligned} \quad (17)$$

The case of special interest here is the case of $\theta = 90°$. Then, Eqs. (17) become

$$\begin{aligned}([K]-\omega^2[M])\{V\} &= \{0\}\\ [C]\{V\} &= \{F\}\end{aligned} \quad (18)$$

Thus, if $\theta = 90°$, i.e., all displacements are in quadrature with the excitation, the response vector will be eigenvector $\{V\}_r$ and the excitation frequency must be a natural frequency ω_r, since Eq. (18a) is the eigenvalue problem associated with Eq. (1). The excitation $\{F\}$ must satisfy Eq. (18b) in order to establish this form of quadrature response.

It may be concluded from the above that, whether the system has damping coupling or not, the following "response criterion" holds:

"For a linear, hysteretically-damped system subject to a set of monophase harmonic forces, a sufficient condition for the response of the system to be in a pure natural mode is that this response be in quadrature with the excitation."

A corollary of this resonance criterion may be derived immediately. Equations (5) and (6) may be combined to give

$$\{\bar{U}\} = [B']\{F\}+i[B'']\{F\} \quad (19)$$

If the response is to be in quadrature with the excitation, then it is required that

$$[B']\{F\} = \{0\} \quad (20)$$

This is a set of homogeneous linear equations which will have a nontrivial solution if, and only if

$$\det([B']) = 0 \tag{21}$$

The corollary may be stated as follows

"The response of a system will be in quadrature with the excitation if, and only if, the determinant of the real part of its complex admittance matrix is equal to zero. The corresponding force ratio required is given by Eq. (20)."

Equations (20) and (21) form the basis of much of the multiple-shaker procedure which will be discussed. Since Asher proposed the above use of the admittance matrix in Ref. 12, it will be referred to as Asher's method.

Multiple-Shaker Test Procedures

The resonance criterion given in the previous section states that if the response of a system is in quadrature with the excitation, then this response constitutes a "pure undamped natural mode" of the system. However, it is neither necessary nor practical to provide a shaker for every degree-of-freedom. But, if the number of shakers, p, is less than the number of degrees of freedom, n, it will not generally be possible to excite the system so that all n responses will be in quadrature with the p forces. This follows from the fact that it will not be possible to satisfy Eq. (20) exactly. Therefore, it becomes necessary to establish criteria defining an acceptable *test natural mode* and to determine how many and which shakers are required to excite such a mode.

Consider an n-degree-of-freedom system (e.g., n may be taken as the total number of motion measurement stations), and let θ_r be the phase angle of the rth element of the response vector $\{v\}$ in Eq. (15). Because 1) the number of shakers employed is usually less than n, 2) the wave generator generally does not allow excitation at the exact natural frequencies, and 3) experimental errors may occur in establishing the force ratios required by Eq. (20), all of the phase angles θ_r $(r = 1, 2, \ldots, n)$ cannot be equal to exactly 90° or 270° simultaneously.

In the following, the "mode shape" obtained from a resonance test will be defined as the quadrature components of the response at all the motion measurement stations. A test mode shape will be termed "acceptable" if the phase angles θ_i satisfy the following "modal purity criteria":

$$\begin{aligned} &|\theta_i - 90°| \leqq \varepsilon(V_i, \omega_r) \\ \text{or}\quad &|\theta_i - 270°| \leqq \varepsilon(V_i, \omega_r) \end{aligned} \tag{22}$$

where ε is the allowable phase error, V_i is the total amplitude of the ith element v_i, and ω_r is the natural frequency of the mode being excited. For example, if, for a certain range of frequency, the allowable quadrature component error is 1% of the largest total amplitude, then the allowable phase error ε is given by Table 1.

Table 1 Example of modal purity criteria

V_i	0.9–1.0	0.8–0.89	0.7–0.79	0.6–0.69	0.5–0.59
ε	5°	6°	7°	8°	9°
V_i	0.4–0.49	0.3–0.39	0.2–0.29	0.1–0.19	0.0–0.09
ε	10°	15°	20°	30°	90°

If an n-degree-of-freedom system is excited by p shakers $(p \leqq n)$ the complex response at the p excitation stations is, from Eq. (5),

$$\{\bar{U}_*\} = [\bar{B}_*]\{F_*\} \tag{23}$$

where $\{F_*\}$ is the vector of the p excitation forces and $[\bar{B}_*]$ is the corresponding $p \times p$ submatrix of $[\bar{B}]$. $\{\bar{U}_*\}$ is the vector of responses at the p excitation stations. Let $[\bar{B}_*]$ be split into real and imaginary parts $[B_*']$ and $[B_*'']$, respectively. It should be noted that each column of $[\bar{B}_*]$ is the frequency-dependent complex (or vector) response at the p exciter stations due to excitation at one of them. That is, each column of $[\bar{B}_*]$ is the result of a single-shaker modal survey with displacements recorded at the p designated shaker locations.

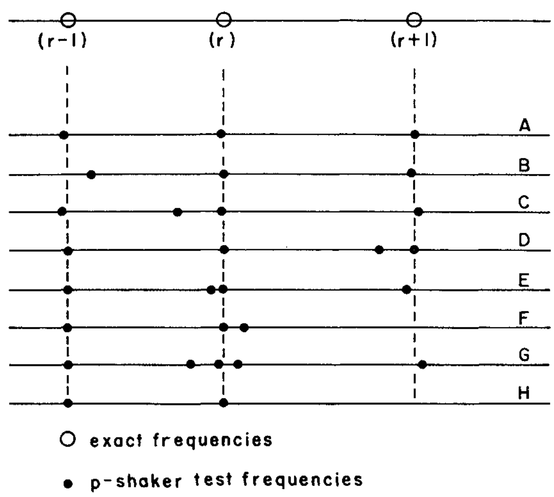


Fig. 1 Typical test natural frequency spectra.

Since $[\bar{B}_*]$ is a function of the excitation frequency, there exists a set of frequencies, ω_*, which satisfy the equation

$$\det([B_*']) = 0 \tag{24}$$

These frequencies will be called "test natural frequencies." They may or may not be the true natural frequencies, which satisfy the corresponding equation, Eq. (21).

If the system is excited by a set of forces at a test natural frequency with the force ratios calculated from

$$[B_*']\{F_*\} = \{0\} \tag{25}$$

then the responses at the exciter stations will, from Eq. (23), be in quadrature with the forces. The responses at the other $(n-p)$ stations where no forces are applied will not necessarily be in quadrature with the excitation. The quadrature components of the response at all n stations, when Eq. (25) is satisfied, constitute a "test natural mode shape." If the test natural mode shape satisfies the modal purity criterion, e.g., Table 1, then the mode will be termed an "acceptable test natural mode shape."

If the frequencies calculated from Eq. (24) are plotted on a frequency line, they will have forms resembling those in Fig. 1. Each line stands for a run, i.e., a solution for roots of Eq. (24), with p shakers. In the vicinity of the rth true natural frequency the admittance matrices $[\bar{B}_*]$ of order $p \times p$ for runs (A) and (B) exhibit the same properties as the true spectrum, i.e., there are frequencies ω_* corresponding to ω_{r-1}, ω_r, and ω_{r+1} but no additional frequencies near ω_r. With the aid of a simulation study it will be shown that the shaker combinations producing frequency lines like (C) through (G), which exhibit randomly-occurring "spurious" frequencies, or (H) which exhibits a missing true frequency, are not likely to produce an acceptable test natural mode at the test natural frequency which approximates ω_r. The following "shaker selection criterion" may be stated:

"A given shaker combination will excite an acceptable test natural mode if its test natural frequency spectrum exhibits the following properties: 1) the frequency, ω_*, under consideration appears on several other spectra; 2) the frequencies in the vicinity of ω_* also appear on several other spectra and are not spurious, i.e., randomly-occurring, frequencies."

A shaker combination which does not produce a spectrum with the preceding properties may produce an acceptable mode, but it will not be as good as the mode obtained if the criteria are met. The simulation study will also show that modes corresponding to spurious frequencies exhibit quite large phase errors at some of the nonexcited stations.

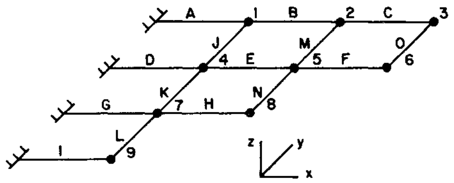


Fig. 2 Nine-degree-of-freedom simulation model.

Simulation Study of a Nine-Degree-of-Freedom Structure

The abovementioned procedure is applied to determine the second and third natural modes of a nine-degree-of-freedom structure. Figure 2 shows its configuration. The structure is assumed to be linear with damping uncoupled. Damping in all modes is assumed to be the same, with $\gamma = 0.02$ in Eq. (10). The structure vibrates in z direction only. Table 2 shows the physical properties of each member of the structure.

The structure is treated as a lumped-mass system with two different sets of masses as given in Table 3. The exact natural frequencies and mode shapes were calculated using the eigenvalue problem associated with Eq. (1). The complete frequency spectra are given in Table 4 and the corresponding second and third mode shapes are given in Table 5. Mass set 2 was designed to give a 3% frequency separation of these modes.

Table 2 System parameters

Beam element	Length (in.)	Area (in.^{2})	I_y(in.^{4}) × 10^{-3}	I_z(in.^{4}) × 10^{-3}	J(in.^{4}) × 10^{-3}
A,D,G,I	8	0.25	5.2	5.2	8.8
B,E,H	8	0.187	2.2	3.9	4.7
C,F	8	0.125	0.65	2.6	1.8
J,K,L	4	0.25	5.2	5.2	8.8
M,N	4	0.187	2.2	3.9	4.7
O	4	0.125	0.65	2.6	1.8

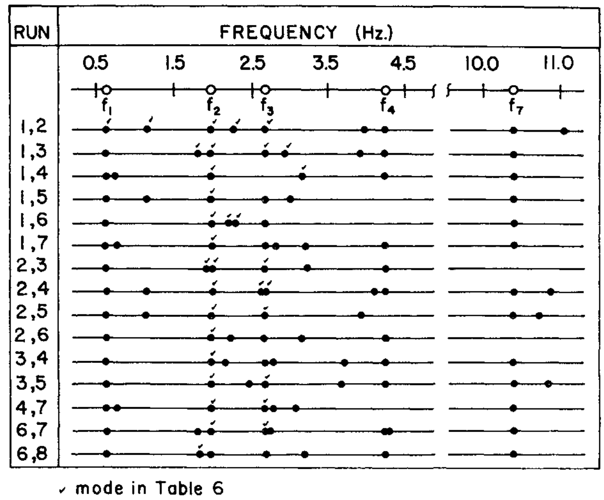


Fig. 3 Frequency spectra for mass set 1, $p = 2$.

Elements $\bar{b}_{ij}$ of the admittance matrix were calculated using all nine terms of the expansion in Eq. (14). Most of the results given below are for $p = 2$. The number of possible shaker combinations, or runs, is thus $9!/(2!\,7!) = 36$ for each mass configuration. Reference 13 contains data for runs not included in this paper. For each run there exists a 2 × 2 admittance matrix $[\bar{B}_*]$. The frequencies, ω_*, which satisfy Eq. (24) are plotted for mass set 1 in Fig. 3. For several selected runs at selected frequencies Table 6 shows the force ratios calculated from Eq. (25) and the corresponding responses calculated from Eq. (5). All computations were carried out on the CDC 6600 computer at The Univ. of Texas at Austin.

From Fig. 3 it may be seen that there are many shaker combinations which satisfy the shaker selection criteria of the first

Table 3 Lumped masses (lb)

Nodal point	1	2	3	4	5	6	7	8	9
Mass set 1	1.0	0.8	0.5	1.0	1.0	0.5	1.0	0.5	0.8
Mass set 2	1.0	1.0	1.5	1.0	1.0	2.0	1.0	0.8	0.8

Table 4 Exact natural frequencies (Hz)

Mode number	1	2	3	4	5	6	7	8	9
Mass set 1	0.617	1.969	2.658	4.213	5.093	6.444	10.38	12.07	22.4
Mass set 2	0.362	1.651	1.703	3.223	4.833	6.214	10.05	11.54	22.4

Table 5 Exact natural modes

Nodal point	1	2	3	4	5	6	7	8	9
Mass set 1									
Second mode	0.360	0.717	−0.959	0.370	0.864	−0.725	0.357	1.000	0.323
Third mode	−0.325	−0.817	−0.720	−0.038	0.056	1.000	0.193	0.814	0.351
Mass set 2									
Second mode	0.216	0.462	−0.628	0.272	0.749	0.099	0.300	1.000	0.292
Third mode	−0.371	−1.00	−0.827	−0.206	−0.446	0.946	−0.069	−0.024	0.022

and second natural frequencies. As for the third mode, although the third natural frequency does not appear on every frequency line, it appears on most of the frequency lines and some of these lines have no spurious frequencies close to the third natural frequency. Therefore, if the proper shakers are used, two shakers may also be used to excite the third mode. Even for the higher

Table 6 Response at test natural frequencies for mass set 1, $p = 2$

Run	Force ratio	Excit. freq.	Total amplitude and phase error $\|\varepsilon\|$ at nodal point								
			1	2	3	4	5	6	7	8	9
1, 2	1.0: −2.00	0.617[a]	0.099	0.409	1.00	0.086	0.371	0.957	0.071	0.329	0.054
			0.0	0.0	0.4[b]	0.1	0.2	0.5	0.3	0.3	0.4
1, 2	1.0: −1.56	1.163	−0.003	−0.011	−0.861	−0.017	−0.122	1.00	−0.033	−0.206	−0.041
			0.0	0.0	85.1	78.3	82.1	85.2	82.9	84.2	84.3
1, 2	1.0: 1.61	1.970[a]	0.360	0.717	0.956	0.370	0.864	−0.726	0.357	1.00	0.323
			0.0	0.0	1.9	2.1	2.8	2.3	3.6	4.4	4.8
1, 2	1.0: 0.82	2.252	0.025	0.053	−0.666	0.175	0.536	−0.198	0.300	1.00	−0.363
			0.0	0.0	84.6	80.9	82.6	69.2	84.4	85.1	85.0
1, 2	1.0: 1.86	2.657[a]	−0.325	−0.818	−0.719	−0.040	0.063	1.00	0.200	0.815	0.352
			0.0	0.0	1.5	14.1	27.5	0.6	4.7	3.8	3.2
1, 3	1.0: 0.54	1.804	0.011	0.297	−0.063	0.154	0.686	−0.928	0.253	1.00	0.293
			0.0	81.1	0.0	79.4	81.6	84.1	80.9	81.9	81.4
1, 3	1.0: 1.57	1.969[a]	0.357	0.712	−0.956	0.368	0.862	−0.727	0.356	1.00	0.323
			0.0	1.3	0.0	1.7	3.4	7.8	3.0	4.5	3.9
1, 3	1.0: −0.04	2.659[a]	−0.325	−0.818	−0.717	−0.039	0.068	1.00	0.199	0.812	0.350
			0.0	3.4	0.0	8.9	33.9	3.4	2.4	2.1	1.4
1, 3	1.0: −0.29	2.901	−0.201	−0.436	−0.010	−0.059	−0.289	1.00	0.090	0.218	0.104
			0.0	80.8	0.0	79.9	84.8	83.6	81.8	78.1	77.4
1, 4	1.0: 18.05	1.970[a]	0.360	0.717	−0.958	0.370	0.864	−0.725	0.357	1.00	0.323
			0.0	1.2	0.6	0.0	1.5	0.1	0.5	1.6	1.0
1, 4	1.0: −0.20	3.146	−0.021	−0.727	−0.595	−0.005	0.219	1.00	0.045	0.195	0.091
			0.0	82.9	84.1	0.0	84.9	83.9	76.4	76.3	75.9
1, 5	1.0: 5.81	1.969[a]	0.360	0.717	−0.960	0.370	0.864	−0.726	0.357	1.00	0.323
			0.0	0.3	1.6	0.0	0.0	1.8	0.1	0.0	0.2
1, 6	1.0: 0.12	1.971[a]	0.359	0.716	−0.954	0.369	0.863	−0.722	0.357	1.00	0.323
			0.0	3.1	2.1	2.9	4.9	0.0	4.9	6.1	6.4
1, 6	1.0: 0.28	2.170	0.027	0.570	−0.183	0.174	0.782	−0.037	0.282	1.00	0.327
			0.0	81.0	69.3	78.2	82.1	0.0	81.8	82.8	83.0
1, 6	1.0: 7.66	2.657[a]	−0.325	−0.817	−0.725	−0.039	0.056	1.00	0.199	0.813	0.351
			0.0	1.1	6.5	6.7	7.6	0.0	2.1	1.7	1.4
1, 7	1.0: 1.46	1.970[a]	0.360	0.717	−0.958	0.370	0.864	−0.725	0.357	1.00	0.323
			0.0	1.3	0.4	0.1	1.3	0.1	0.0	1.3	0.0
2, 3	1.0: 0.80	1.924	−0.055	−0.105	0.184	−0.145	−0.497	1.00	−0.233	−0.876	−0.273
			1.1	0.0	0.0	60.8	68.6	79.7	69.7	73.4	72.7
2, 3	1.0: 1.03	1.967[a]	0.335	0.666	−0.908	0.351	0.832	−0.780	0.347	1.00	0.322
			0.2	0.0	0.0	7.4	11.8	28.8	13.0	17.8	16.9
2, 3	1.0: 0.32	2.658[a]	−0.325	−0.817	−0.719	−0.039	0.059	1.00	0.199	0.814	0.351
			0.4	0.0	0.0	9.3	17.7	1.5	2.6	2.2	1.6
2, 4	1.0: 3.60	1.969[a]	0.360	0.718	−0.958	0.370	0.864	−0.726	0.357	1.00	0.323
			0.6	0.0	0.2	0.0	1.1	1.8	0.8	1.8	1.6
2, 4	1.0: −65.91	2.636	−0.216	−0.467	−0.774	−0.029	0.468	0.625	0.180	1.00	0.319
			27.1	0.0	56.0	0.0	82.5	22.6	51.1	59.9	50.0
2, 4	1.0: 109.9	2.644[a]	−0.275	−0.645	−0.799	−0.036	0.409	0.819	0.197	1.00	0.349
			18.1	0.0	43.5	0.0	80.3	14.8	38.0	48.3	37.1
2, 5	1.0: 5.27	1.969[a]	0.360	0.717	−0.959	0.370	0.864	−0.726	0.357	1.00	0.323
			0.0	0.0	1.6	0.0	0.0	2.2	0.2	0.2	0.4
2, 5	1.0: −0.95	2.659[a]	−0.325	−0.817	−0.720	−0.038	0.056	1.00	0.199	0.814	0.351
			0.1	0.0	1.4	0.9	0.0	0.6	0.8	0.3	0.8
2, 6	1.0: −0.56	1.970[a]	0.359	0.717	−0.959	0.370	0.863	−0.725	0.357	1.00	0.323
			1.1	0.0	5.2	2.9	3.1	0.0	4.4	5.0	5.5
3, 4	1.0: 7.21	1.970[a]	0.360	0.717	−0.957	0.370	0.864	−0.727	0.357	1.00	0.323
			0.2	2.0	0.0	0.0	2.8	3.8	1.0	3.2	2.0
3, 5	1.0: −1.67	1.969[a]	0.360	0.716	−0.959	0.370	0.864	−0.725	0.357	1.00	0.323
			0.8	0.7	0.0	0.4	0.0	2.1	0.2	0.3	0.1
3, 5	1.0: 0.39	2.656[a]	−0.321	−0.807	−0.712	−0.038	0.056	1.00	0.197	0.804	0.347
			3.9	3.3	0.0	11.6	0.0	8.7	1.1	2.7	1.8
4, 7	1.0: 0.30	1.970[a]	0.360	0.717	−0.958	0.370	0.864	−0.725	0.357	1.00	0.323
			0.5	1.6	0.4	0.0	1.3	0.2	0.0	1.2	0.2
4, 7	1.0: −0.66	2.660[a]	−0.325	−0.818	−0.718	−0.038	0.059	1.00	0.198	0.810	0.350
			2.6	5.0	3.6	0.0	18.5	5.0	0.0	1.5	0.1
6, 7	1.0: −0.45	1.969[a]	0.361	0.721	−0.963	0.370	0.865	−0.728	0.357	1.00	0.323
			2.1	3.5	5.1	0.9	1.0	0.0	0.0	0.5	0.7
6, 7	1.0: −8.82	2.660[a]	−0.327	−0.823	−0.775	−0.042	0.163	1.00	0.203	0.844	0.357
			5.6	5.2	19.7	28.6	65.3	0.0	0.0	13.1	1.4
6, 8	1.0: 0.61	1.827	−0.286	−0.825	1.00	−0.152	−0.392	0.041	−0.037	−0.008	−0.040
			82.5	83.1	84.7	81.8	82.6	0.0	77.3	0.0	82.4

[a] Approximately equal to a natural frequency.
[b] Phase error given in Degrees.

modes two shakers may be sufficient to excite an acceptable mode if the frequency separation is great enough.

It can be seen in Fig. 3 that the frequencies calculated from the condition of $\det([B_*'])=0$ are not necessarily the true natural frequencies. On the other hand, the true natural frequencies may or may not be able to satisfy the above condition. However, the nine true natural frequencies appear repeatedly on almost all of the frequency lines in Fig. 3 and these frequency lines are based on two shakers only. Thus, if a frequency which satisfies the condition of $\det([B_*'])=0$ occurs in several of the runs, then this frequency most likely is a natural frequency of the system.

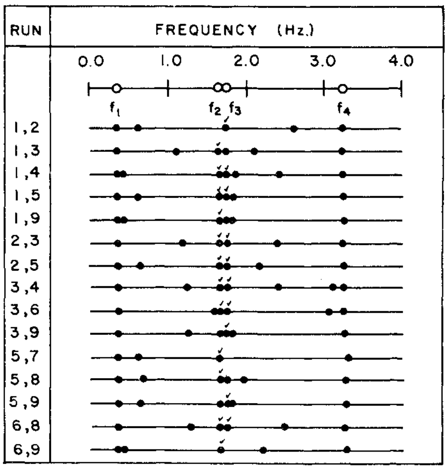
From the responses shown in Table 6 it can be seen that, if the structure is excited by a pair of forces with frequencies and force ratios calculated from the condition $[B_*']\{F_*\}=0$, the responses of the two excitation stations will be in simple harmonic motion but 90° out of phase with the forces. Further, if the excitation frequency is a close approximation to one of the true natural frequencies the response at the other seven stations, calculated from Eq. (5), will also be in harmonic motion and nearly $\pm 90°$ out of phase with the excitation forces, e.g., (1, 2) at 1.970 Hz and (1, 3) at 1.969 Hz. If the excitation frequency is a spurious frequency the responses of the other seven stations will generally have phase errors of 65° or more, e.g., (1, 2) at 2.252 Hz and (1, 3) at 1.804 Hz. This phenomenon can thus be applied to confirm the natural frequencies.

If the frequency spectra in Fig. 3 are compared on the basis of the calculated phase errors in Table 6, it is seen that shaker combinations whose spectra satisfy the shaker selection criterion stated previously do produce the best mode shapes. As an example consider the shaker combination (1, 4) at 1.970 Hz. The mode produce is very good with a maximum phase error of 1.6°.

To determine whether the abovementioned conclusion remains valid when the system has closely-spaced true natural frequencies the second mass set was employed. The true natural frequencies are given in Table 4. The second and third natural frequencies have a separation of 3% in this case. The test natural frequencies calculated from Eq. (24) are plotted in Fig. 4, while the responses and corresponding force ratios are shown in Table 7. From Fig. 4 it is seen that again most runs had frequencies corresponding to the true second and third natural frequencies and that spurious frequencies, while more numerous than in the previous case, occurred in a quite random manner. From Table 7 it can be seen that the test natural mode shapes obtained are not quite as good as in the previous case but that the conclusion stated above is still valid. For example, shaker combinations (2, 3), (2, 5), (3, 4), etc., for which spurious frequencies are not close to the second and third natural frequencies, will lead to good approxi-

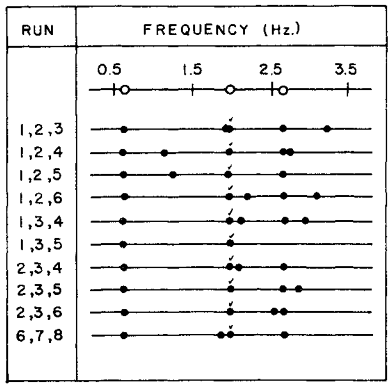
Table 7 Response at test natural frequencies for mass set 2, $p = 2$

Run	Force ratio	Excit. freq.	Total amplitude and phase error $\lvert\varepsilon\rvert$ at nodal point 1	2	3	4	5	6	7	8	9
1, 2	1.0: 0.06	1.700	−0.375	−1.00	−0.780	−0.299	−0.527	0.906	−0.126	−0.334	−0.097
			0.0	0.0	20.3	11.7	17.7	9.4	38.7	65.1	80.9
1, 3	1.0: −0.52	1.652	0.214	0.456	−0.634	0.271	0.747	0.109	0.299	1.00	0.292
			0.0	1.6	0.0	0.7	1.0	14.6	1.0	1.0	1.2
1, 4	1.0: −2.18	1.652	0.220	0.473	−0.620	0.275	0.754	0.097	0.300	1.00	0.292
			0.0	0.6	5.3	0.0	1.7	24.6	1.0	2.6	1.8
1, 4	1.0: −0.94	1.805	−0.369	−1.00	−0.884	−0.199	−0.426	0.794	−0.060	0.092	0.038
			0.0	3.6	12.6	0.0	3.0	8.4	9.3	78.0	28.2
1, 5	1.0: −1.10	1.652	0.220	0.473	−0.619	0.274	0.754	0.098	0.300	1.00	0.292
			0.0	2.1	3.1	0.5	0.0	23.8	0.9	0.9	1.2
1, 5	1.0: −0.35	1.705	−0.369	−1.00	−0.876	−0.200	−0.427	0.972	−0.063	0.073	0.040
			0.0	5.0	12.4	2.6	0.0	9.4	16.6	78.8	35.8
1, 9	1.0: 3.76	1.652	0.220	0.474	−0.620	0.275	0.754	0.098	0.300	1.00	0.292
			0.0	0.2	6.0	1.3	2.3	26.3	1.2	3.1	0.0
2, 3	1.0: −1.30	1.651	0.215	0.459	−0.631	0.272	0.748	0.105	0.300	1.00	0.292
			0.3	0.0	0.0	0.5	0.3	11.1	0.6	0.5	0.7
2, 3	1.0: 0.65	1.702	−0.371	−1.00	−0.821	−0.207	−0.449	0.943	−0.071	−0.040	0.022
			0.1	0.0	0.0	1.2	1.8	1.4	5.6	40.8	20.8
2, 5	1.0: −2.41	1.651	0.217	0.465	−0.625	0.273	0.751	0.097	0.300	1.00	0.292
			0.1	0.0	0.8	0.1	0.0	8.6	0.2	0.2	0.3
2, 5	1.0: −0.65	1.703	−0.371	−1.00	−0.833	−0.205	−0.442	0.949	−0.067	−0.017	0.024
			0.0	0.0	2.9	0.3	0.0	1.4	0.1	4.0	0.1
3, 4	1.0: −0.346	1.651	0.215	0.458	−0.632	0.272	0.748	0.105	0.299	1.00	0.292
			0.4	1.4	0.0	0.0	0.7	9.4	0.1	0.5	0.3
3, 4	1.0: 2.50	1.702	−0.371	−1.00	−0.820	−0.207	−0.450	0.943	−0.071	−0.041	0.020
			0.1	0.9	0.0	0.0	2.6	1.7	1.6	39.4	10.4
3, 6	1.0: 1.14	1.648	0.234	0.515	−0.592	0.281	0.770	0.073	0.302	1.00	0.290
			18.5	21.1	0.0	14.6	14.5	0.0	12.3	11.8	10.8
3, 6	1.0: −15.2	1.702	−0.370	−1.00	−0.849	−0.203	−0.439	0.957	−0.068	−0.086	0.038
			4.0	3.7	0.0	7.4	9.5	0.0	21.9	85.3	37.3
3, 9	1.0: 3.12	1.705	−0.375	−1.00	−0.748	−0.222	−0.507	0.891	−0.092	−0.192	0.004
			14.2	13.1	0.0	18.1	22.7	2.9	24.6	62.5	0.0
5, 7	1.0: 5.37	1.652	0.224	0.483	−0.615	0.276	0.757	0.102	0.300	1.00	0.292
			3.3	4.3	6.0	1.1	0.0	36.2	0.0	1.6	0.8
5, 8	1.0: 6.53	1.651	0.218	0.466	−0.624	0.273	0.751	0.097	0.300	1.00	0.292
			0.2	0.4	1.2	0.0	0.0	11.2	0.1	0.0	0.2
5, 9	1.0: −3.72	1.716	−0.292	−0.818	−1.00	−0.137	−0.271	0.939	−0.026	0.293	0.043
			25.5	28.0	45.3	16.7	0.0	38.2	14.1	73.0	0.0
6, 8	1.0: 13.20	1.651	0.215	0.459	−0.630	0.272	0.748	0.102	0.299	1.00	0.292
			1.9	2.2	0.4	0.9	0.7	0.0	0.4	0.0	0.0
6, 8	1.0: −0.12	1.702	−0.371	−1.00	−0.824	−0.207	−0.448	0.944	−0.070	−0.028	0.021
			0.7	1.0	3.4	0.5	0.9	0.0	0.0	0.0	2.9
7, 9	1.0: 0.13	1.652	0.220	0.473	−0.620	0.274	0.754	0.097	0.300	1.00	0.292
			0.7	1.0	4.5	0.0	1.2	22.8	0.0	1.8	0.0



✓ mode in Table 7

Fig. 4 Frequency spectra for mass set 2, $p = 2$.



✓ mode in Table 8

Fig. 5 Frequency spectra for mass set 1, $p = 3$.

mations to the second mode. As might be expected, the combination (6, 8) produces very good second and third modes because each of these points is almost on a node line of one mode while it has a large amplitude in the other mode.

To assess the effect on frequencies and mode shapes resulting from the addition of another shaker, and to establish guidelines for selecting the optimum location for the added shaker, a limited series of three-shaker runs was conducted. The frequency spectra are shown in Fig. 5, while the responses and force ratios are given in Table 8. The results are for mass set 1.

From the limited data available, two strategies for adding a shaker appear promising. They are: 1) add the shaker at the location whose amplitude error is greatest, i.e., a point with large phase error and large total amplitude; and 2) add the shaker at a location that reinforces the desired mode but is near a node line of the interfering mode.

The first strategy employs all of the shakers, the last shaker added being used to cancel the dominant errors from the previous shakers. For example, suppose the initially-chosen shaker locations were (2, 3). From Fig. 3 it is seen that there is a spurious frequency which interferes with the second frequency, and from Table 6 it is seen that the resulting mode shape has large phase errors at locations with large amplitude, e.g., nodal point 6. A shaker added at 6 produces the frequency line (2, 3, 6) of Fig. 5, and the corresponding mode shape from Table 8 is seen to be excellent.

The second strategy is basically one of replacement. It can be illustrated best on the data from mass set 2. Consider an initial test with shaker combination (3, 6). From Table 7 it is seen that neither mode 2 nor mode 3 is acceptable. Figure 4 indicates the presence of a spurious frequency which leads to the poor modes. Although neither mode is acceptable, the modal data do indicate that point 8, which is near a mode line of mode 3 but is the largest-amplitude point of mode 2, would be preferable to point 3 as an excitation point. Table 7 confirms that shaker combination (6, 8) is indeed an excellent choice for exciting both modes.

It should also be mentioned that in some cases replacement of a shaker produces better results than adding a shaker. For example, the mode shape given in Table 6 for (2, 3) at 1.967 Hz is not acceptable. If a shaker were added at 4 the mode given in Table 8 for (2, 3, 4) is seen to be acceptable. However, this three-shaker mode is not quite as good as the two-shaker mode for (2, 4) at 1.969 Hz given in Table 6.

It may be noted that no mention has been made of the use of the mass matrix either in adjusting the shaker force ratios or in testing for purity of modes using orthogonality criteria. Experience has shown that the mass matrix, which must be obtained analytically, may be subject to considerable error.

Table 8 Response at test natural frequencies for mass set 1, $p = 3$

Run	Force ratios	Excit. freq.	Total amplitude and phase error $\|\varepsilon\|$ at nodal point 1	2	3	4	5	6	7	8	9
1, 2, 3	1.0: −22.00: −22.00	1.967	0.335	0.667	−0.909	0.351	0.832	−0.781	0.348	1.00	0.322
			0.0	0.0	0.0	7.5	11.8	28.8	13.1	17.8	16.9
1, 2, 4	1.0: −0.8 : −2.4	1.969	0.360	0.717	−0.958	0.370	0.864	−0.726	0.357	1.00	0.322
			0.0	0.0	0.0	0.0	0.9	2.0	0.5	1.5	1.1
1, 2, 5	1.0: −3.0 : −14.00	1.969	0.360	0.717	0.959	0.370	0.864	0.726	0.357	1.00	0.323
			0.0	0.0	1.6	0.0	0.0	2.2	0.2	0.2	0.4
1, 2, 6	1.0: 0.42: −0.38	1.970	0.360	0.717	−0.959	0.370	0.863	0.726	0.357	1.00	0.323
			0.0	0.0	4.6	2.1	2.8	0.0	3.8	4.5	5.0
1, 3, 4	1.0: −3.6 : −28.00	1.970	0.360	0.717	−0.957	0.370	0.864	−0.727	0.357	1.00	0.323
			0.0	2.0	0.0	0.0	2.7	3.5	0.9	3.1	1.8
1, 3, 5	1.0: −0.8 : 1.2	1.969	0.360	0.717	−0.960	0.370	0.864	−0.725	0.357	1.00	0.323
			0.0	0.2	0.0	0.1	0.0	1.3	0.2	0.0	0.0
2, 3, 4	1.0: 0.26: 3.70	1.970	0.360	0.717	−0.958	0.370	0.864	−0.726	0.357	1.00	0.323
			0.7	0.0	0.0	0.0	1.4	2.9	1.0	2.2	1.9
2, 3, 5	1.0: −1.3 : 1.7	1.969	0.360	0.717	−0.959	0.370	0.864	−0.725	0.357	1.00	0.323
			0.2	0.0	0.0	0.3	0.0	1.0	0.4	0.1	0.5
2, 3, 6	1.0: −0.42: 0.24	1.970	0.360	0.717	0.958	0.370	0.864	0.725	0.357	1.00	0.323
			0.3	0.0	0.0	1.1	1.1	0.0	1.7	1.9	2.3
6, 7, 8	1.0: −0.09: −0.25	1.969	0.360	0.717	−0.958	0.370	0.864	−0.725	0.357	1.00	0.323
			1.7	2.8	4.0	0.7	1.0	0.0	0.0	0.0	0.0

Hence, in the present discussion modal purity has been based on phase coherence of the excited modes rather than on orthogonality of the generalized mass matrix. The latter, in fact, may be used to locate and correct errors in the analytical mass matrix.

Based on the assumption that the damping forces are proportional to the elastic forces, Lewis and Wrisley[5] state that "each one of the numerous oscillating forces acting on a complex structure at a natural frequency must be adjusted in magnitude to be proportional to the product of the mass of structure on which it acts and the amplitude of that mass in the mode being excited." The present simulation study does not indicate that this is a valid criterion when only a few of the mass points are used as shaker locations. For example, Table 6 shows that a very good second mode is excited at 1.970 Hz by shaker combination (1, 4) with the force ratio 1.0 : 18.05. The force ratio based on product of masses (Table 3) and amplitudes would be 1.0 : 1.03. Numerous other examples can be drawn to show that adjusting force ratios according to the criterion of Lewis and Wrisley is not a valid strategy.

Summary and Conclusions

The present paper has employed a simulation study to explore procedures for conducting multiple-shaker resonance tests for determining natural mode shapes. It has been assumed that single-shaker resonance tests would be conducted initially, and where these did not suffice to define accurate natural frequencies and mode shapes the vector (co-quad) response data from these tests would be available for use in conducting multiple-shaker tests. A procedure, suggested by Asher, for using the admittance matrix based on a small number of shaker locations has been examined. It has been shown that calculations using this admittance matrix data can be employed to predict which shaker combination is most likely to produce acceptable mode shape data and to determine the proper ratio of shaker forces to employ. Shaker selection criteria are given, and their validity is tested through use of a simulation study of a nine-degree-of-freedom damped structure. Since close spacing of natural frequencies adds to the difficulty of obtaining accurate mode shape data in a resonance test, two models were studied, one with 35% minimum frequency spacing and one with 3% spacing.

In recently-conducted modal survey tests increasing use has been made of vector response measurements and of hybrid computers techniques for control of tests and data acquisition and reduction. This makes it feasible to incorporate the procedures suggested in this paper into the control logic using calculations based on Eqs. (24) and (25) to determine shaker locations and shaker force ratios. The proposed method appears particularly useful when close frequency spacing makes it difficult to obtain accurate mode shape data otherwise.

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